James Stewart Calculus

Derivative

The idea of the derivative is a very simple idea. There is nothing complicated about it.
Typically to learn the derivative, you need to read and learn hundreds of pages of some calculus book and solve many exercises but that is all detail and not that important provided that you know your basics well.
The idea of the derivative itself can be presented using a simple example and formalized just in a few pages in such a way that anybody can understand it and appreciate its power and beauty.
The basic concepts that you need to learn the derivative easily are,

- functions

- limit
- continuity

It is needless to menthon that to understand functions, you need to understand the basic ideas in mathematics like,

- algebra
- Trigonometry
- geometry
- numbers
- simple arithmetic
- how to prove simple results in general
- etc.

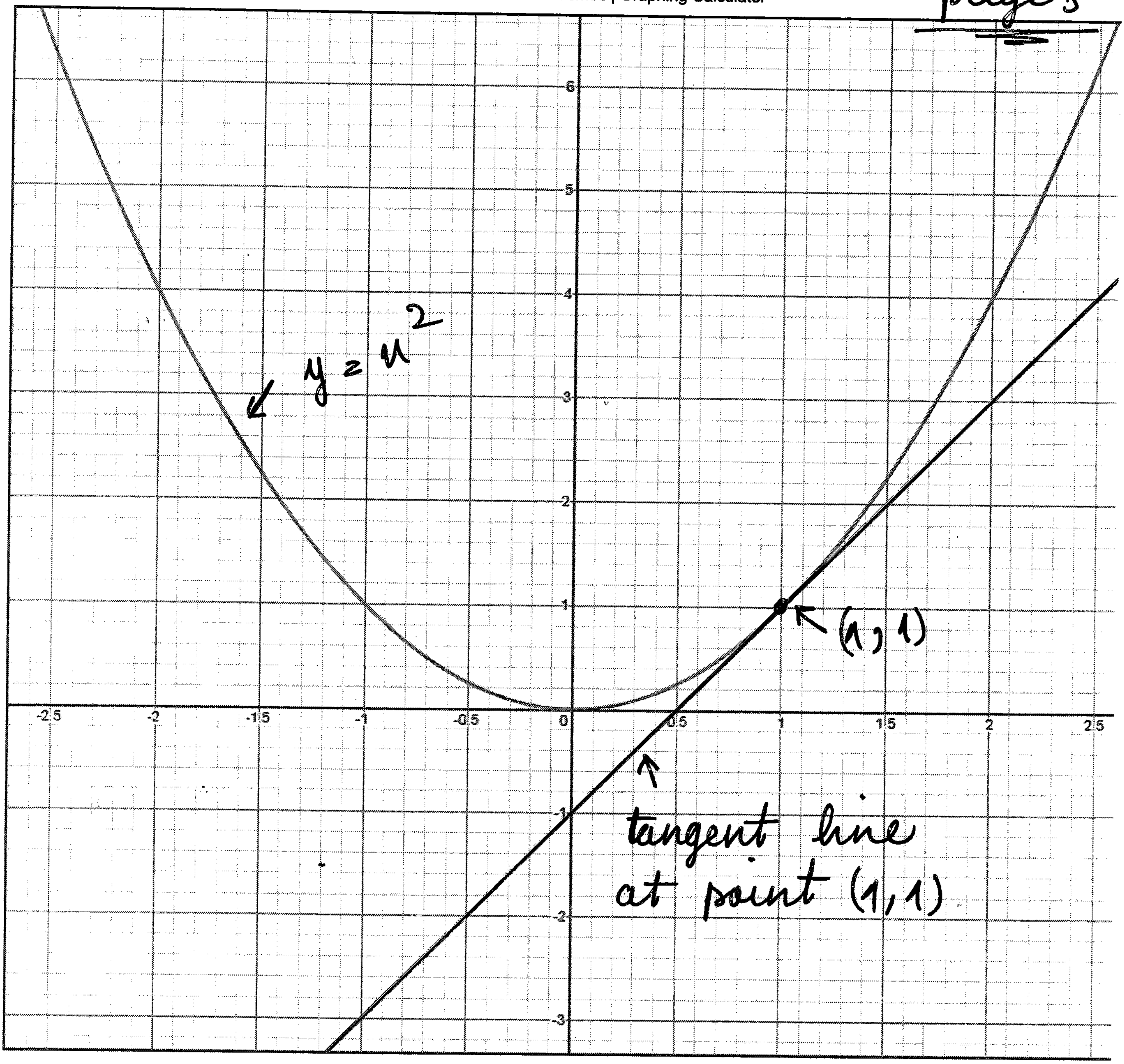
In simple words, you need to understand the mathematics you have learned from class 6 all the way up to say class 10 or 11.

It is also important to note that if you don't have enough mastery over the basic ideas, it might be very difficult for you, or anybody else for that matter, to understand the derivative. The idea of the derivative or the whole concept of calculus is nothing new in mathematics apart from the basics of mathematics like ~~g~~ geometry and algebra. Once mathematicians could understand the basics well, calculus was born. In other words, calculus is nothing but using the same old algebra, geometry, trigonometry, etc. in more clever ways.

In this course, therefore, you are expected to understand your basic mathematics well. If that is not the case, or if you have doubts, you can always review your basics at least and spend some time on the concepts that need practice. Moreover make sure you have understood

limits and continuity that come before the derivative.
with those things out of the ways, we'll start with the derivative.
We'll start with an example and later try to state everything formally.
Whenever you calculate the derivative of a function at some point, you are calculating the slope of a line tangent to the graph of the function at that point. If this does not tell you much, please bear with me. Things will be clear shortly.
Regarding the tangent line consider the example of the function $f(x) = x^2$ and the tangent line to the graph of this function at the point (1, 1) as shown in the graph on the next page.

page 5



1. $f(x) = x^2$
2. $(1,1)$
3. $y - 1 = 2(x - 1)$

In the graph on page 5, assuming that the straight line is tangent to the graph of the function at the point (1,1), then the value of the slope of this line is called the value of the derivative of the function $f(x) = x^2$ at $x = 1$.

Assuming that the slope of the line is 2, we can say that the derivative of the function $f(x) = x^2$ at $x = 1$ is equal to 2. Symbolically, this statement can be written as,

$$\frac{d}{dx}\left(f(x)\right)\Big|_{x=1} = 2$$

OR $$\frac{d}{dx}\left(x^2\right)\Big|_{x=1} = 2$$

Now, why is it at all important or useful to be able to find the slope of this straight line?

It is important because otherwise you could not tell how fast you were driving in your car at any moment in time. In other words, your speed or velocity would always be unknown.
In turn, if you had no way to calculate your velocity right now, you would not know how fast you had to drive to get to your meeting on time.
And by the same logic, all technology that is dependent upon such calculations as velocity like the satellite would not be possible.
If the satellite is not there, then your GPS device does not work. And you see where this is going.
In simple words, all the technology we have in the world today is, in one way or another, dependent upon the slope of

this straight line.
Now, let us see why the slope of this straight line can be taken as the velocity of your car at any moment.
This was one of the basic problems for the sake of which the limit and the derivative were discovered in physics and mathematics: the problem of motion.
For centuries and millennia people could not understand the concept of motion.
The main problem was that people could calculate average velocity very easily but instantaneous velocity could not be calculated.
Suppose that the motion of your car is described by some function like $f(x) = x^2$. In other words, f gives the position of your car at any moment x.

For example at time $x = x_1$ you are at $f(x_1)$ and at time $x = x_2$, you are at $f(x_2)$.

To calculate your average velocity between the times x_1 and x_2, you need to know how much change there was in your position which is simply $f(x_2) - f(x_1)$. You also need to know the elapsed time for this change to take place which is $x_2 - x_1$. Now, if you take the ratio of these two, this, by the definition of average velocity is your average velocity between the times x_1 and x_2, i.e.,

$$\left(\begin{array}{l}\text{Average velocity} \\ \text{between } x_1 \text{ and} \\ x_2\end{array}\right) = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Since some time has gone by, $x_2 - x_1 \neq 0$. And so you are not dividing by zero. Moreover, assuming that ~~some~~ there was

some change in the position,
then $f(x_2) - f(x_1) \neq 0$, which means
that, $\frac{f(x_2) - f(x_1)}{x_2 - x_1}$ is going to be some
real number.

Now, this is all good if all you wanted
to calculate was "average" velocity.

Now suppose that you want to calculate
your velocity exactly at $x = x_1$. ~~we kn~~
Then at $x = x_1$, no time has gone by
and since no time has gone by, ~~you kno~~
there has been no change in the position.

Therefore, $\frac{\text{change in position}}{\text{change in time}} = 0/0$

$(0/0)$ is the indeterminate form in
mathematics, which means you cannot
calculate it.

This problem took mathematicians and
physicists over 2000 years until finally