Exercise 12: If a ball is thrown into the air with a velocity of 40 ft/s, its height (in feet) after t seconds is given by $y = 40t - 16t^2$. Find the velocity when $t = 2$.

Solution: $f(t) = 40t - 16t^2$ is the position function of the ball.

We know that velocity is the derivative of position. Therefore to find the velocity at $t = 2$, all you need to do is to find $f'(a)$ based on the definition of the derivative and then in that expression, set $a = 2$.

If you do the calculation correctly you should come up with,

$$f'(a) = -32a + 40$$

And $f'(2) = -32(2) + 40$

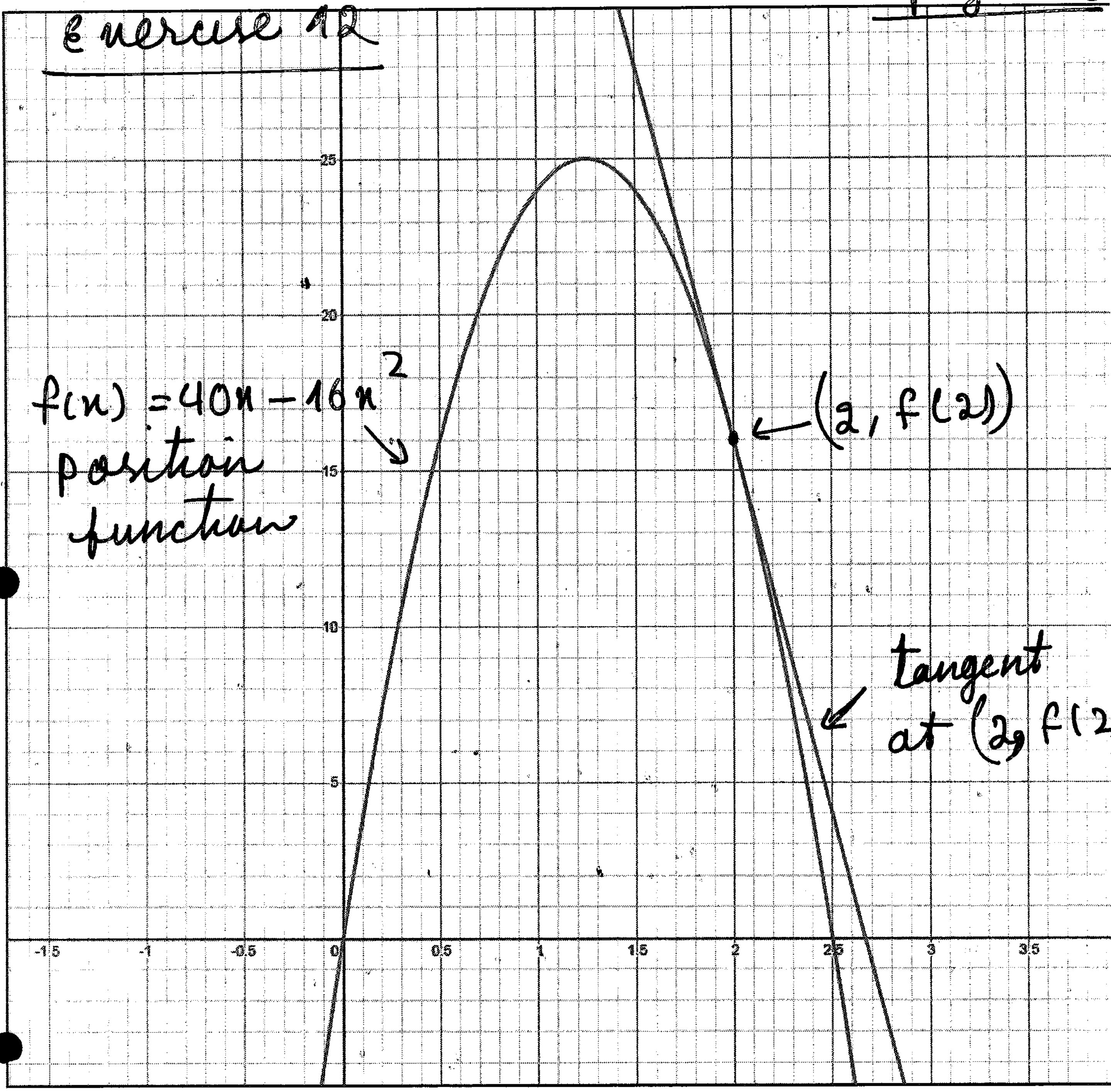
$= -64 + 40 = -24$ feet/s

Therefore the velocity at time $t = 2s = -24$ ft/s.

The velocity at $t = 2$ is negative because at $t = 1.25$ s, the ball reaches its max. height and starts to come back towards the earth, meaning that at $t = 2s$, the ball is moving in the negative direction based on our coordinate system.

You can see a graph of f and the tangent line at $x = 2$ on page 133.

You can see that at $x = 2$, the tangent line has a negative slope equal to -24 exactly. The slope of the tangent line at this point is the value of the derivative of the function at this point or the instantaneous rate of change of position WRT time which is called velocity by definition.



1 $f(x) = 40x - 16x^2$

2 $(2, f(2))$

3 $y - f(2) = -24(x - 2)$

Exercise 13 :

Consider the function $y = f(x) = 1/x^2$.

(a) Find the slope of the tangent line to the graph of f at point $x = a$.

(b) Find the derivative of f at $x = a$.

(c) If f represented the position function of a particle moving along a straight line, y measured in meters and x measured in seconds, then what would be the instantaneous velocity of the particle at time $x = 5$ s ?

What would be the average velocity of the particle between the times $x = 2$ s and $x = 7$ s ?

Solution : To find the slope of the tangent line to the graph of the function $f(x) = 1/x^2$ at the point $x = a$, we can use the definition, ~~of~~

$$m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

We have,

$f(x) = 1/x^2$

$f(a+h) = \dfrac{1}{(a+h)^2}$; $f(a) = \dfrac{1}{a^2}$

$$m = \lim_{h \to 0} \frac{\frac{1}{(a+h)^2} - \frac{1}{a^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{a^2 - (a+h)^2}{(a+h)^2 a^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{a^2 - (a^2 + 2ah + h^2)}{(a+h)^2 a^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{a^2} - \cancel{a^2} - 2ah - h^2}{(a+h)^2 a^2}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{\cancel{h}(-2a-h)}{(a+h)^2 a^2}}{\cancel{h}}$$

$$= \lim_{h\to 0} \frac{-2a-h}{(a+h)^2 a^2}$$

$$= \frac{-2a}{a^2 a^2} = \frac{-2a}{a^4} = \frac{-2}{a^3}$$

$\therefore \boxed{m = \frac{-2}{a^3}}$ ← slope of the tangent line to the curve $y = 1/x^2$ at $x = a$

(b) To find the derivative of f at $x = a$, we use the def.,

$$f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

But this def. is exactly the same as the def. we used for the tangent line in the previous step.

Therefore we can conclude that, ~~the~~

– the slope of the tangent line to the ~~on~~ graph of the function ~~of~~ f at point $x = a$, and

- the derivative of f at

$x = a$

are one and the same thing. It is nothing but the rate at which y is changing WRT x if $f(x) = y$. (More on this in the next exercise). This can be calculated as

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(c) Now, if f represented the position function of particle moving along a straight line ($y = f(x)$), then y would represent position in meters and x would represent time in seconds (given in the problem).

We know that velocity by definition, the rate at which position changes WRT.

On the other hand, we mentioned before that the derivative of f at 'a' is the

rate at which y changes
WRT x if $y = f(x)$.
Therefore if we calculate the derivative of f at $x = a$, we have, in effect, calculated the velocity at $x = a$.
Also if we calculate the slope of the tangent to the curve $y = f(x)$ at $x = a$, that again is the same thing as the velocity at $x = a$. This is because, as we mentioned before, the slope of the tangent line at 'a' is the same as the derivative ~~of~~ at 'a' both for the function f.

Now, since in part (a), we calculated,

$m = -2/a^3$, we can call this $f'(a)$ meaning the derivative of f at 'a' or we can also call it $v(a)$ meaning velocity at $x = a$ if the function f represents the position function of a particle moving along a straight line.

Therefore, $v(a) = -2/a^3$.

Now to calculate velocity at $x = 5\,s$, we calculate,

$$v(5) = -2/5^3 = -2/125 \; m/s = 0.016 \; m/s.$$

The unit for velocity comes out as m/s because we made the assumption that distance should be measured in meters and time in seconds.

Moreover, the negative value for velocity means that at $x = 5\,s$, the particle is moving backwards (in the opposite direction of the positive direction in our coordinate system.

Now to calculate the average velocity between the times $x = 2\,s$ and $x = 5\,s$, what we can do is ~~the following:~~ ask the following questions:

- where were we at $x = 2\,s$?
- where did we end up at $x = 5\,s$?