Limits

Definition 1 We write,

$$\lim_{x \to a} f(x) = L$$

and say, "the limit of f(x), as x approaches a, equals L",

if we can make the values of f(x) arbitrarily close to L (as close to L as we like) by taking x to be sufficiently close to a (on either side of a) but not equal to a.

Roughly speaking, this says that the values of f(x) tend to get closer and closer to the number L as x get closer and closer to number 'a' (from either side of a) but $x \neq a$.

An alternative notation for,

$$\lim_{x \to a} f(x) = L$$

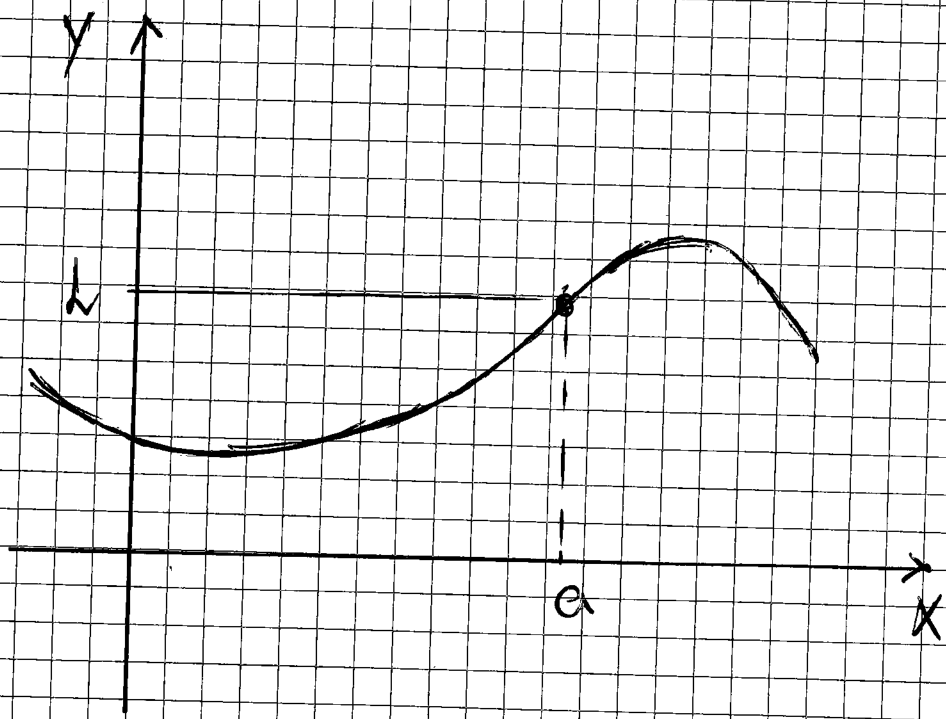
is $f(x) \longrightarrow L$ as $x \longrightarrow a$

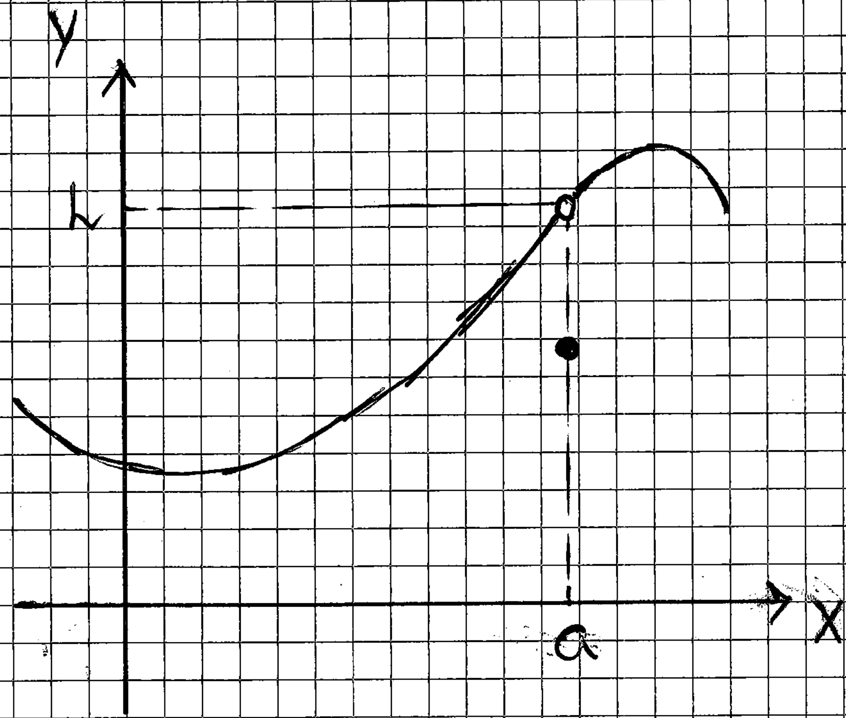
which is read as,

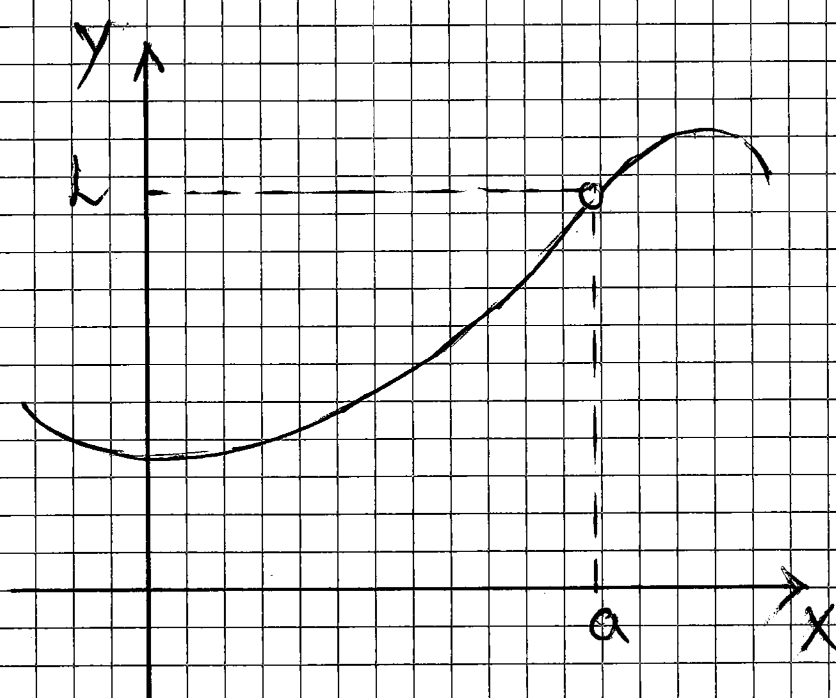
f(x) approaches L as x approaches a.

The phrase $x \neq a$ in the definition means that in finding the limit of $f(x)$ as x approaches a, we never consider $x = a$. In fact, $f(x)$ need not even be defined at $x = a$. The only thing that matters is how f is defined near a.

In the following diagrams, in each case, $\lim_{x \to a} f(x) = L$.







Definition 2 : Left-hand and Right-hand Limits

We write,

$$\lim_{x \to a^-} f(x) = L$$

and say the left-hand limit of f(x)

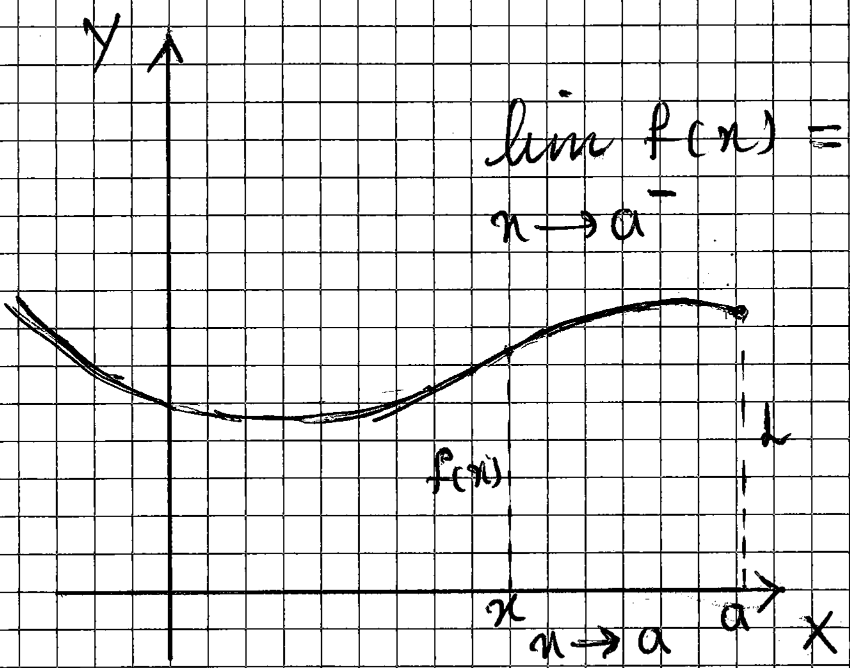
as x approaches 'a' (or the limit of $f(x)$ as x approaches a from the left) is equal to L if we can make the values of $f(x)$ arbitrarily close to L by taking x to be sufficiently close to 'a' and $x < a$.

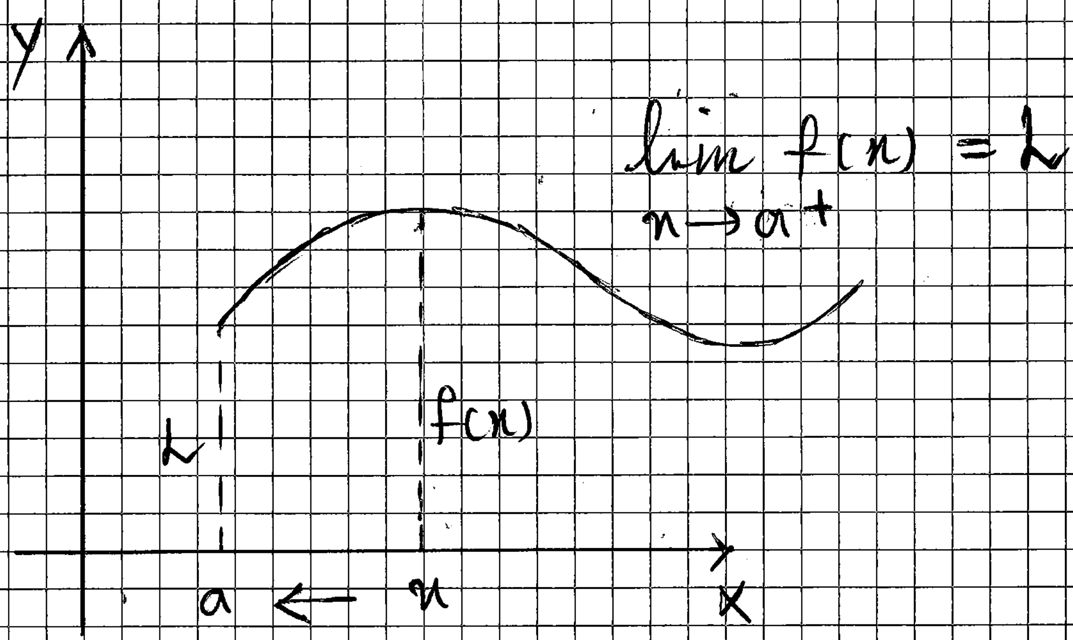
Notice that Definition 2 differs from Definition 1 only in that we require x to be less than a.

Similarly, if we require that x be greater than 'a', we get the right-hand limit of $f(x)$ as x approaches a is equal to L and we write,

$$\lim_{x \to a^+} f(x) = L$$

The symbol $x \to a^+$ means that we consider only $x > a$. These definitions are illustrated in the following diagrams :





By comparing Definition 1 with the definition of one-sided limits, we see that the following is true:

Definition 3:

$$\lim_{x \to a} f(x) = L \iff \lim_{x \to a^-} f(x) = L \text{ and } \lim_{x \to a^+} f(x) = L.$$

Definition 4 : Infinity Limit

Let f be a function defined on both sides of 'a', except possibly at 'a' itself.

Then, $\lim_{x \to a} f(x) = \infty$

means that the values of $f(x)$ can be made arbitrarily large (as large as we please) by taking x sufficiently close to a, but not equal to 'a'.

Another notation for $\lim_{x \to a} f(x) = \infty$ is,

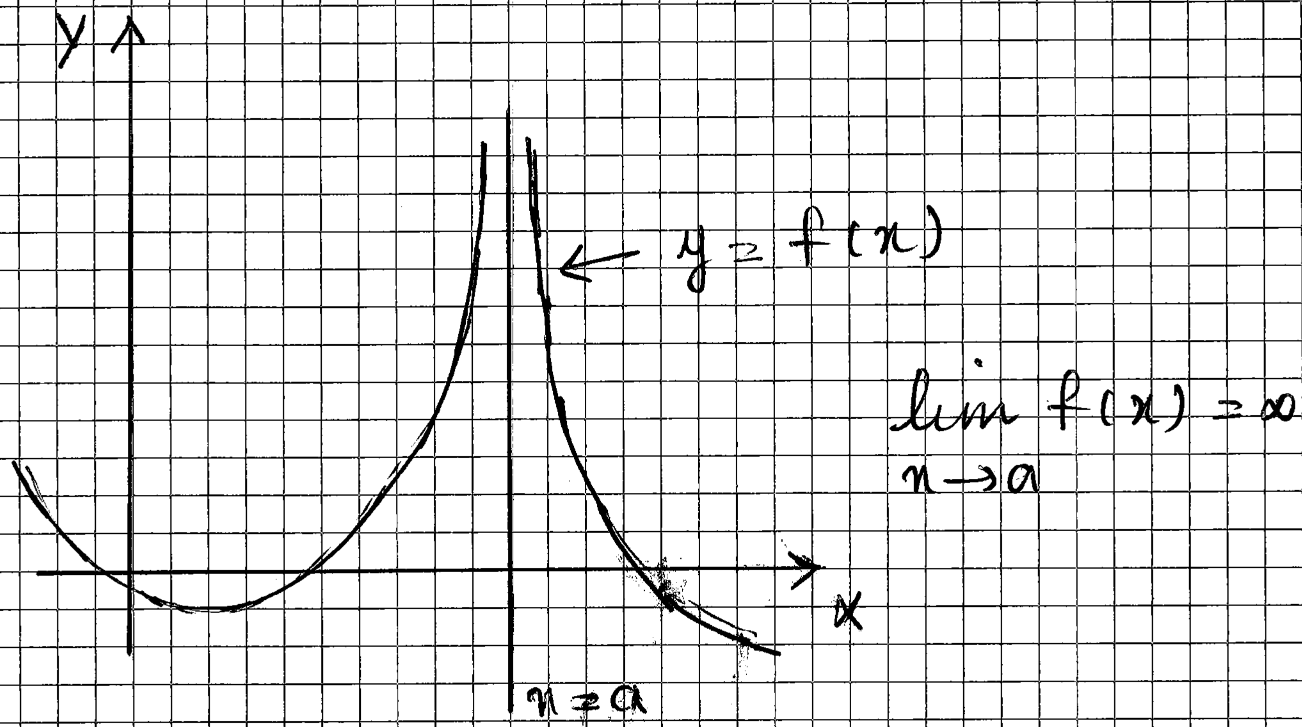
$f(x) \longrightarrow \infty$ as $x \longrightarrow a$

The symbol ∞ is not a number, but the expression $\lim_{x \to a} f(x) = \infty$ is often read as,

the limit of $f(x)$, as x approaches 'a' is infinity.

OR, $f(x)$ becomes infinite as x approaches 'a'.

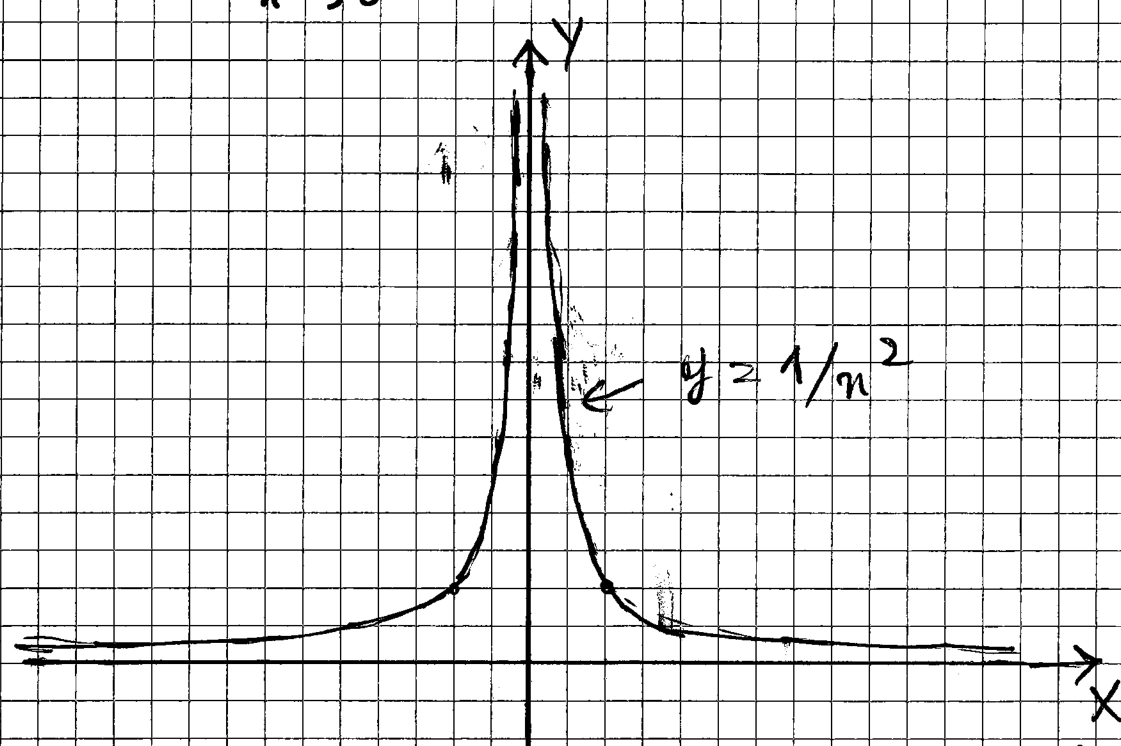
OR, f(x) increases without bound as x approaches 'a'.
This definition is illustrated graphically in the diagram below:



To understand definition 6, consider the following example.
We wish to find $\lim_{x \to 0} \left(\frac{1}{x^2}\right)$ if it exists.
As x gets closer and closer to zero, x^2 get closer and closer to zero and $1/x^2$ becomes very large. Based on the graph of the function on the next page, you can see

that the values of $f(x)$ can be made arbitrarily large by taking x close enough to 0.

Since the values of $f(x)$ do not approach a number, $\lim_{x \to 0} (1/x^2)$ does not exist.



To indicate the kind of behavior exhibited in this function around $x = 0$, we use the notation,

$$\lim_{x \to 0} (1/x^2) = \infty$$

This does not mean that we are regarding ∞ as a number, nor does it mean that the limit exists.

It simply expresses the particular way in which the limit does not exist: $1/x^2$ can be made as large as we like by taking x close enough to 0.

In general, we write symbolically,

$$\lim_{x \to a} f(x) = \infty$$

to indicate that the values of $f(x)$ tend to become larger and larger (or increase without bound) as x gets closer and closer to 'a'.

Definition 5: Negative Infinity Limit

Let f be defined on both sides of 'a', except possibly at 'a' itself. Then,

$$\lim_{x \to a} f(x) = -\infty$$

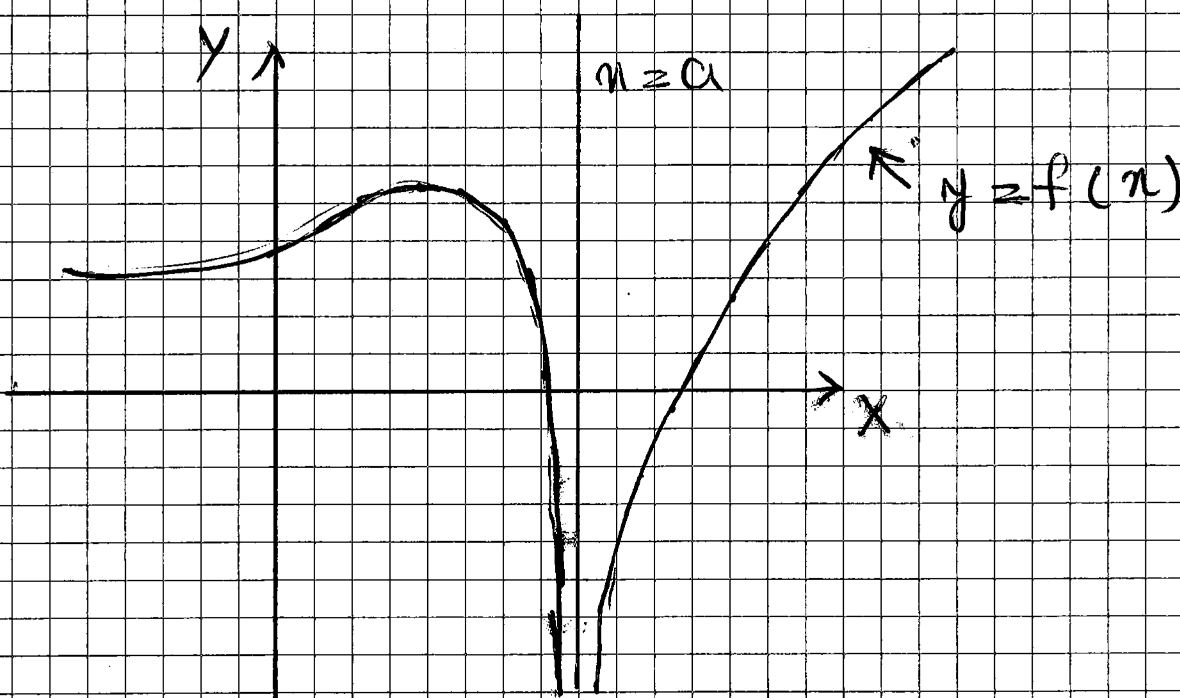
means that the values of $f(x)$ can be made arbitrarily large negative by taking x sufficiently close to 'a' but not equal to 'a'.

The symbol $\lim_{x \to a} f(x) = -\infty$ can be read as,

the limit of f(x), as x approaches 'a', is negative infinity, or

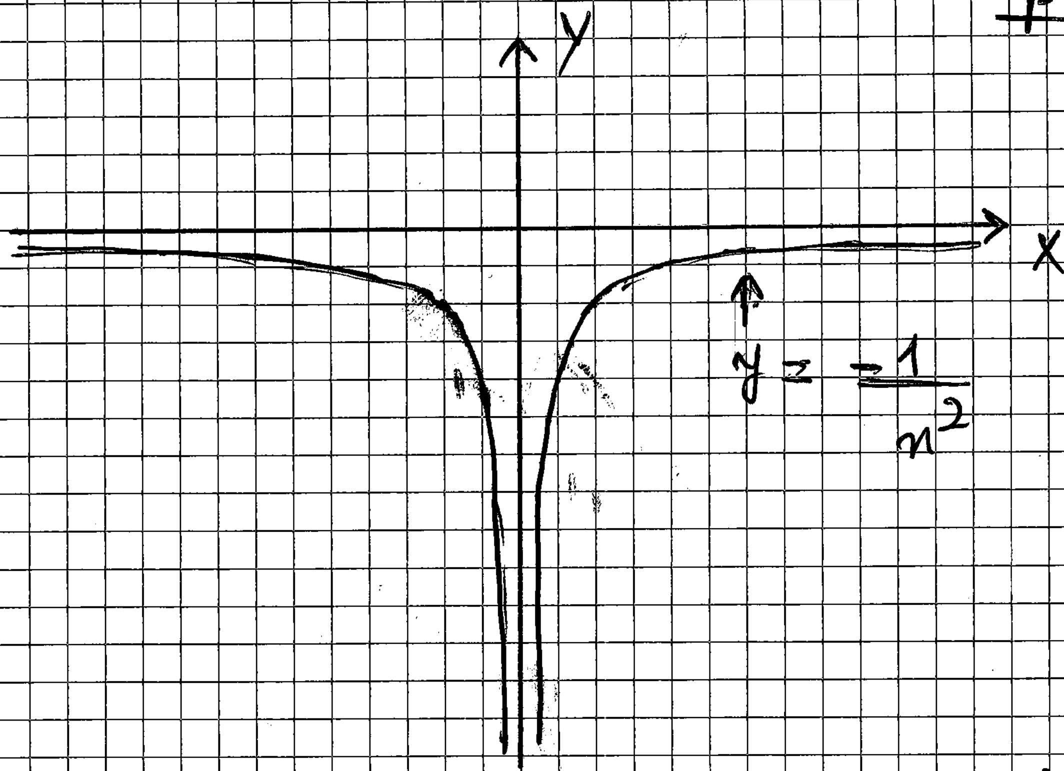
f(x) decreases without bound as x approaches 'a'.

The following diagram illustrates this definition:



To mention an example for this definition we can consider the limit,

$$\lim_{x \to 0} \left(-\frac{1}{x^2} \right) = -\infty$$



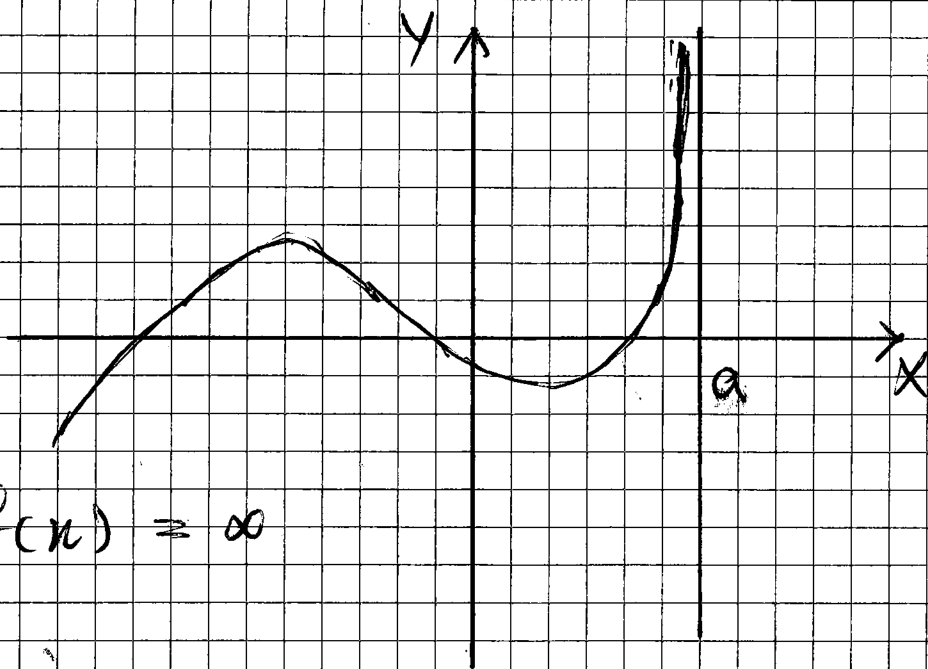
Definition 5-1 : One-Sided Positive and Negative Infinity Limits

With regards to definitions 4 and 5, similar definitions can be given for one-sided infinite limits :

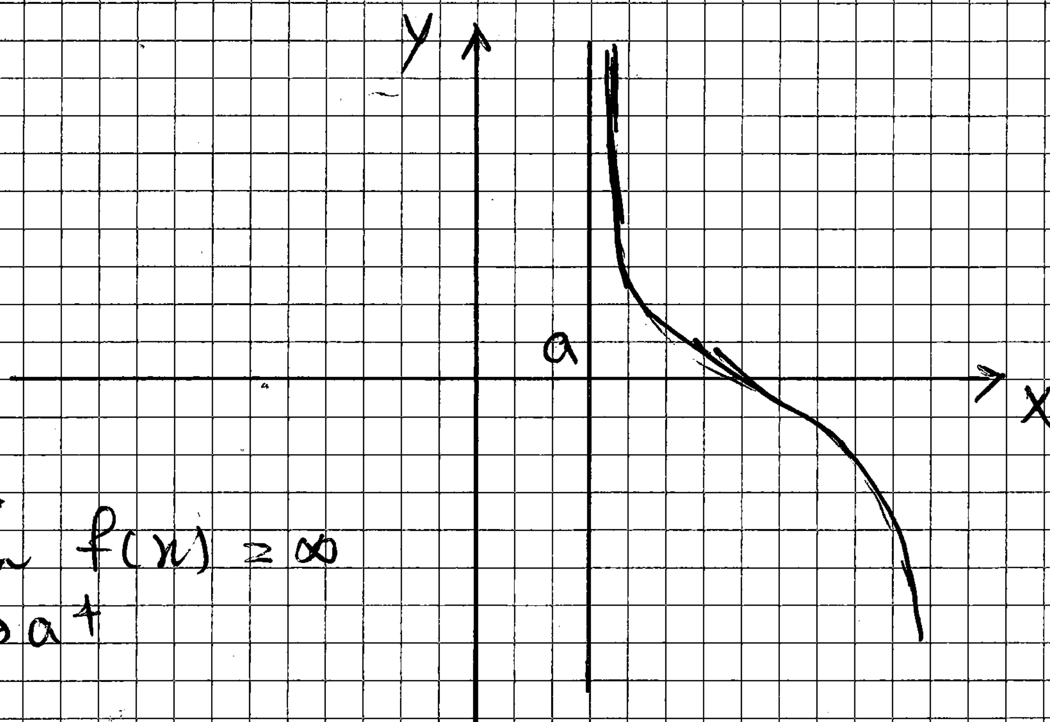
$$\lim_{x \to a^-} f(x) = \infty \qquad \lim_{x \to a^-} f(x) = -\infty$$

$$\lim_{x \to a^+} f(x) = \infty \qquad \lim_{x \to a^+} f(x) = -\infty$$

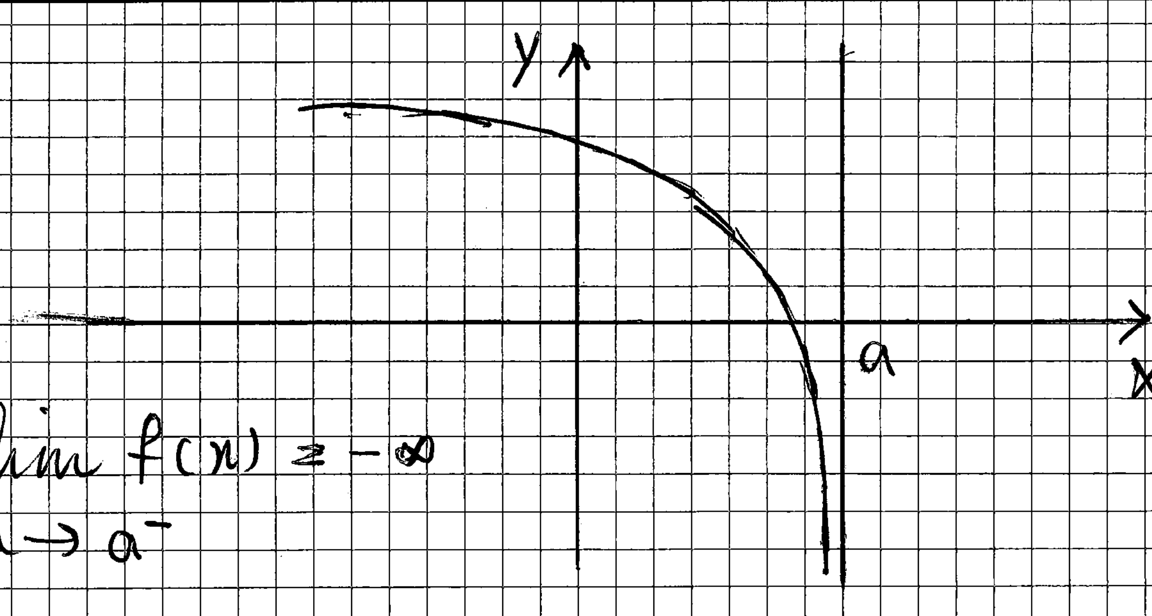
These four definitions are illustrated in the following four diagrams :



$$\lim_{x \to a^-} f(x) = \infty$$



$$\lim_{x \to a^+} f(x) = \infty$$



$$\lim_{x \to a^-} f(x) = -\infty$$