Continuity

We know that the limit of a function as x approaches 'a' can often be found simply by calculating the value of the function at 'a'.

Functions with this property are called continuous at 'a'.

A continuous process is one that takes place gradually, without interruption or abrupt changes.

<u>Definition 1</u>: A function f is continuous at a number 'a' if,

$$\lim_{x \to a} f(x) = f(a)$$

Notice that the definition implicitly requires three things if f is continuous at a:

(i) $f(a)$ is defined, i.e., a is in the domain of f.

(ii) $\lim_{x \to a} f(x)$ exists

(iii) $\lim_{x \to a} f(x) = f(a)$

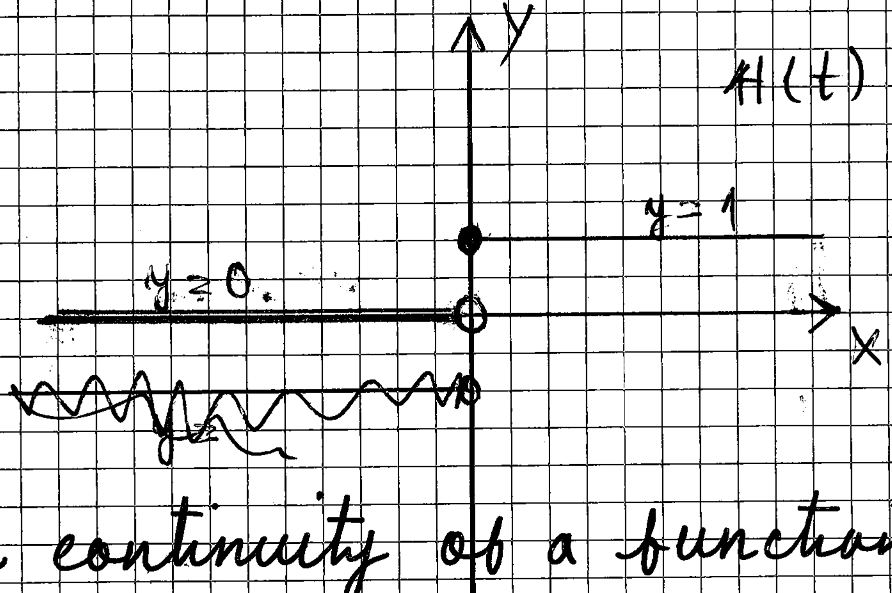
The definition says that f is continuous at 'a' if f(x) approaches f(a) as x approaches 'a'.

Thus a continuous function has the property that a small change in x produces only a small change in f(x). In fact, the change in f(x) can be kept as small as we please by keeping the change in x sufficiently small.

If f is defined near 'a', in other words, if f is defined on an open interval containing 'a', except perhaps at 'a', we say that f is discontinuous at 'a' or f has a discontinuity at 'a' if f is not continuous at 'a'.

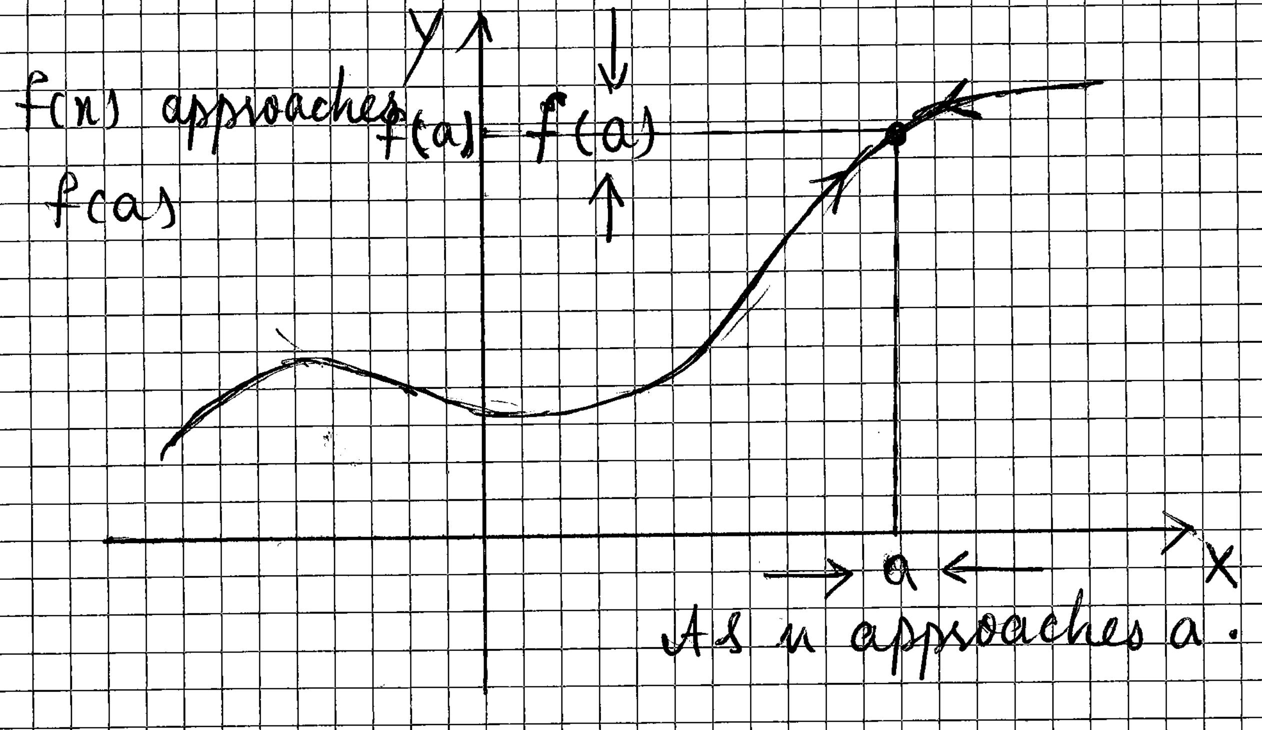
Physical phenomena are usually continuous. For instance, the displacement or velocity of a vehicle varies continuously with time, as does a person's height.

But discontinuities do occur in such situations as electric currents. An example would be the Heaviside function a graph of which we provide below:



$$H(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geq 0 \end{cases}$$

The continuity of a function is illustrated in the following diagram:



As illustrated in the diagram, if f is continuous, then the point $(x, f(x))$ on the graph of f approaches the point $(a, f(a))$ on the graph. So there is no gap in the curve.

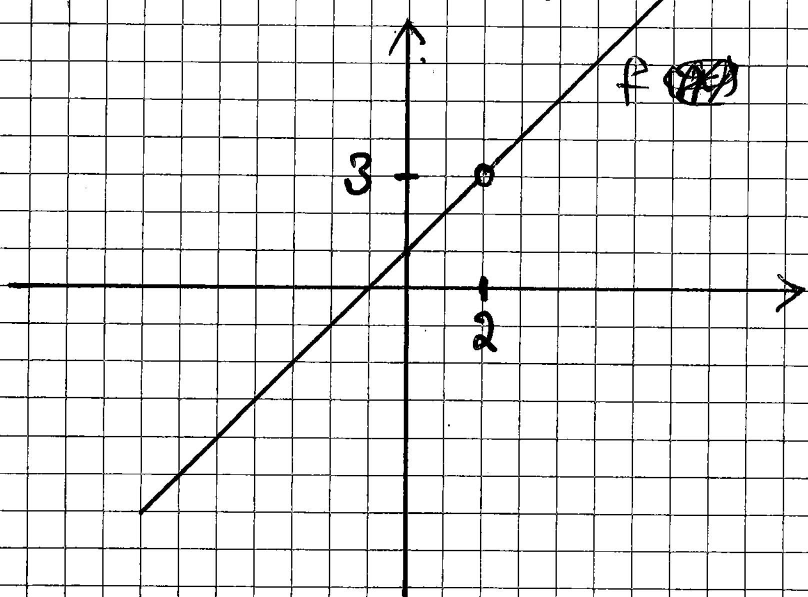
As some examples for studying discontinuities in functions, consider the following examples:

In each of the following cases, identify where each function is discontinuous?

<u>Example 1</u>: $f(x) = \dfrac{x^2 - x - 2}{x - 2}$

This function is not defined at $x = 2$. Therefore it is discontinuous at $x = 2$

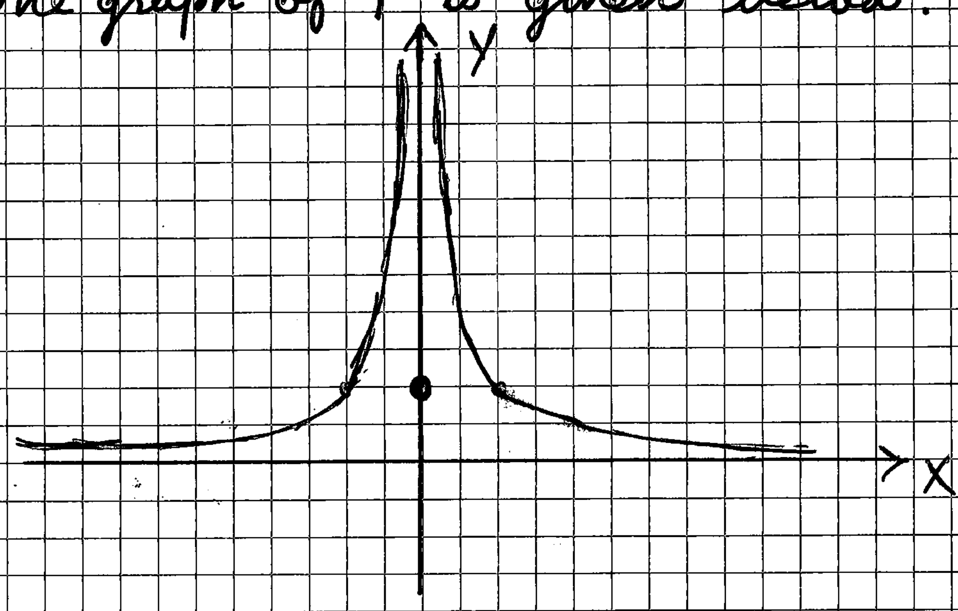
Below, we'll provide the graph of this function.



As you can see there is hole in the graph of the function at $(2,3)$. This is an example of a removable discontinuity because we could remove the discontinuity by redefining f at the single number $x=2$.

<u>Example 2</u> : $f(x) = \begin{cases} 1/x^2 & , x \neq 0 \\ 1 & , x = 0 \end{cases}$

<u>Solution</u>: In the case of this function, $f(0)$ is defined but $\lim_{x \to 0} f(x) = \infty$ which means that the limit is not defined. Therefore f is not continuous at $x=0$. This is an example of an infinite discontinuity. The graph of f is given below:



Example 3:

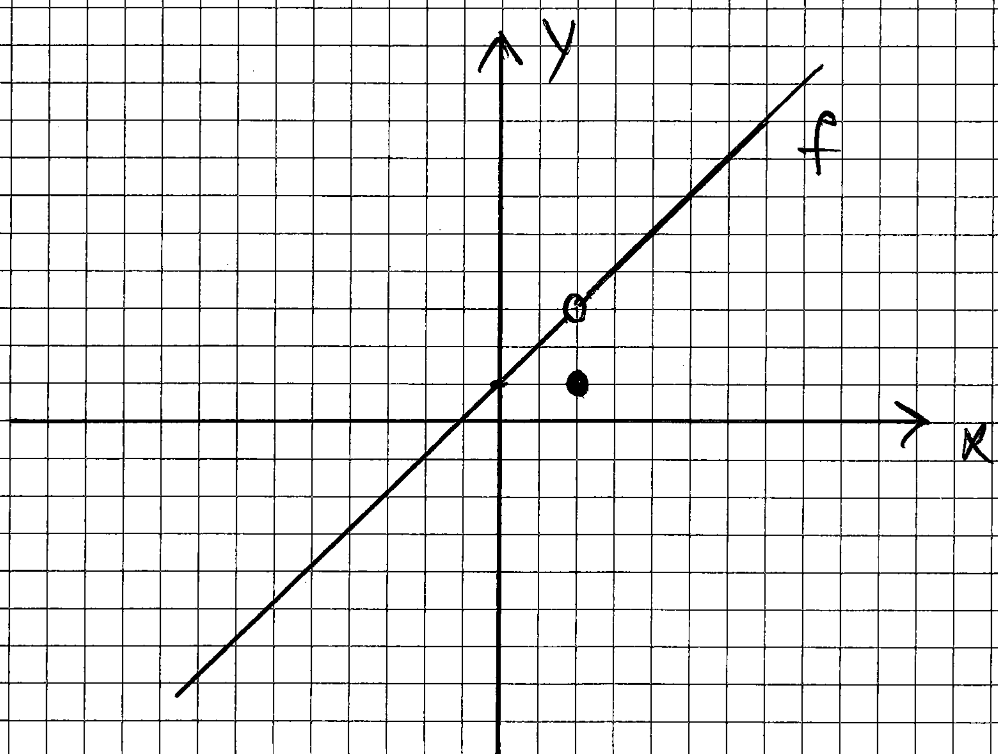
$$f(x) = \begin{cases} \dfrac{x^2 - x - 2}{x - 2}, & x \neq 2 \\ 1, & x = 2 \end{cases}$$

Solution: In the case of this function,

$\lim_{x \to 2} f(x) = 3$ but $f(2) = 1$. Therefore,

$\lim_{x \to 2} f(x) \neq f(2)$. Therefore f is not continuous at $x = 2$.

This is an example of removable discontinuity at $x = 2$. The graph of f is given below:

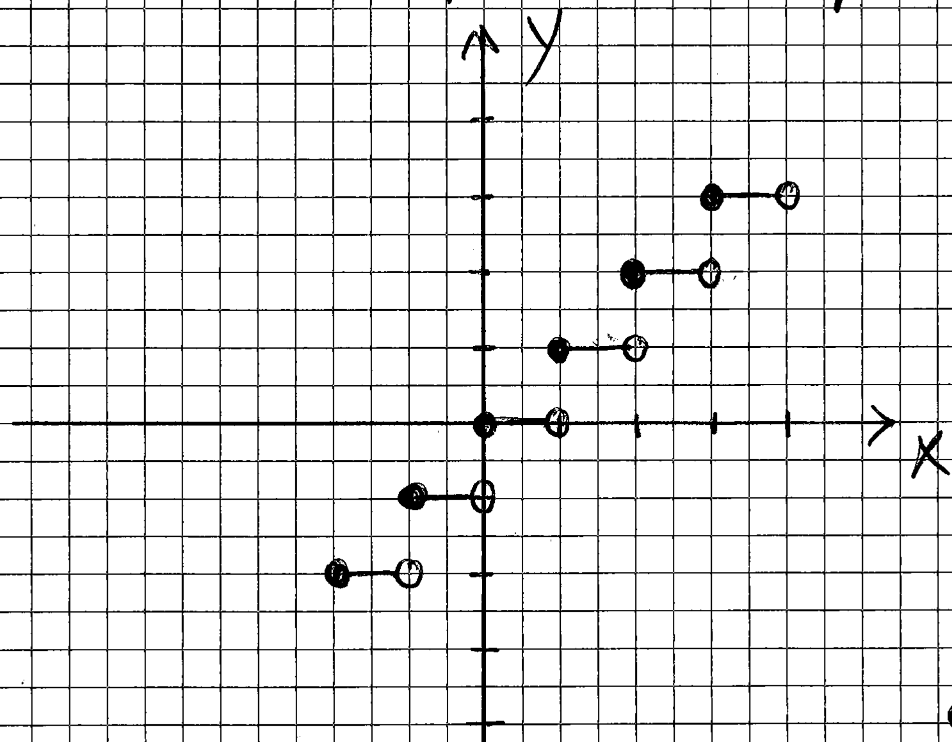


Example 4: $f(x) = [x]$

Solution: The greatest integer function has discontinuities at all integers because $\lim_{x \to n} [x]$, $n \in \mathbb{Z}$ does not exist.

The graph of the function is given below.

This is an example of jump discontinuity.



Definition 2: Continuous from the Right and Left

A function f is continuous from the right at a number 'a' if

$$\lim_{x \to a^+} f(x) = f(a)$$

and f is continuous from the left at 'a' if

$$\lim_{x \to a^-} f(x) = f(a)$$

A good example for this definition would be the greatest integer function a graph of which is given in example 4.

For the greatest integer function we have,

$$\lim_{x \to n^+} f(x) = \lim_{x \to n^+} [x] = n = f(n)$$

$$\lim_{x \to n^-} f(x) = \lim_{x \to n^-} [x] = n-1 \neq f(n)$$

Therefore, at each integer, the greatest integer function is continuous from the right but not continuous from the left.

Definition 3: Continuity on an Interval.

A function f is continuous on an interval if it is continuous at every number in the interval.

If f is defined only on one side of an endpoint of the interval, we understand continuous at the endpoint to mean continuous from the right or from the left.

The following example makes this definition clear.

Example 5: show that the function $f(x) = 1 - \sqrt{1-x^2}$ is continuous on the interval $[-1, 1]$.

Solution: Let $a \in (-1, 1)$. We have,

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(1 - \sqrt{1-x^2}\right)$$

$$= 1 - \lim_{x \to a} \sqrt{1-x^2}$$

$$= 1 - \sqrt{\lim_{x \to a} (1-x^2)}$$

$$= 1 - \sqrt{1-a^2} = f(a)$$

$\therefore$ when $a \in (-1, 1)$,

$$\lim_{x \to a} f(x) = f(a)$$

$\therefore$ f is continuous on the interval $(-1, 1)$.

Similar calculations show that,

$$\lim_{x \to -1^+} f(x) = 1 = f(-1)$$

$$\lim_{x \to 1^-} f(x) = 1 = f(1)$$

Therefore f is continuous from the right at -1 and from the left at 1.

Therefore f is continuous on the interval $[-1, 1]$.

Definition 4 :

- If f and g are continuous at 'a' and 'c' is a constant, then the following functions are also continuous at 'a':

(i) $f+g$ (iv) fg

(ii) $f-g$ (v) f/g, $g(a) \neq 0$.

(iii) cf

- Proof : Each of the five parts of this theorem follows from the corresponding limit laws. We'll provide the proof for part (i).

Since f and g are continuous at 'a',

- we have,

$$\lim_{x \to a} f(x) = f(a) \quad \text{and} \quad \lim_{x \to a} g(x) = g(a)$$

$$\lim_{x \to a} \big((f+g)(x)\big) = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$$

$$= f(a) + g(a)$$

- $$= (f+g)(a)$$